1. **TEXT AND OTHER MATERIALS:**
   - **Check Learning Resources** in **shared class files**
   - Calculus Wiki-book: https://en.wikibooks.org/wiki/Calculus (Main Reference e-Book)
   - Paul's Online Math Notes: http://tutorial.math.lamar.edu
   - Calculus. Early Transcendental Functions by Larson & Edwards, **5th** Editions
   - Calculus. Early Transcendental Functions by Larson & Edwards, **6th** Editions
   - http://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/textbook/

2. Tutorial: http://archives.math.utk.edu/visual.calculus/
   Tutorial, Animation: http://www2.latech.edu/~schroder/animations.htm
   Tutorial: https://www.math.ucdavis.edu/~kouba/ProblemsList.html
   Tutorial: http://www.straighterline.com/landing/online-calculus-video-tutorials/#.Vb_en_lVhBc

3. **Technology Resources**:
   - Desmos Graphic Calculator at https://www.desmos.com/calculator

4. Web based resources
   - Khan academy at: http://www.**khanacademy**.org
     - Exercises and videos on Limits and derivatives: https://www.khanacademy.org/exercisedashboard
   - You tube at: http://www.youtube.com
   - **Google** at: http://www.google.com
5. Calculus I - Practice Problems
   - Paul's Online Math Notes: http://tutorial.math.lamar.edu/problems/calci/calci.aspx

Chapter 2: Page

Limit and Their Properties

General objectives

In this chapter you should be able to:

- Compare Calculus with Precalculus
- Find limits graphically and numerically
- Evaluate limits analytically
- Determine continuity at a point and on an open interval
- Determine one – sided limits
- Determine infinite limits and find vertical asymptotes.

Back Ground

Important Ideas Assumed:

- **Numbers**
- **Sets**
- **Operations**
- **Expressions**
- **Equations**
- **Inequalities**
- **Solutions and solution sets**
- **Constants**
- **Variables: dependent and independent**
- **x – y coordinates, x – y plane**
- **Relations**
- **Domain and range**
- **Functions: polynomial, rational, square root, absolute value, exponential, logarithmic, trigonometric**
- **Graphs**

2.1 A Preview of Calculus

Objectives
In this section you should be able to:

- Understand what Calculus is and how it compares with Precalculus
- Understand that the tangent problem is basic to calculus
- Understand that the area problem is also basic to calculus

What is Calculus?

1. **Calculus is the mathematics of change**
   Change?
   Velocity (instantaneous not average)
   Acceleration
   Tangent lines (Arbitrary Curves)
   Slopes (Arbitrary curves)
   Areas, Volumes (Irregular shaped figures and Objects)
   Arc lengths
   Etc.
2. **Calculus is a branch of mathematics that deals with limits, differentiation and integration**

How about Precalculus Mathematics?
We have:
Velocity (Average)
Acceleration (Constant Rate of Change)
Tangent lines (For Circles)
Slopes (For Straight Lines)
Areas, Volumes, (For Regular Shaped figures, objects)

Calculus and Precalculus are tied by the *idea of limit*.

$$\textit{Precalculus} \rightarrow \textit{Limit Process} \rightarrow \textit{Calculus}$$

The Tangent Line and the Area Problems

These are two historically important classical problems. These two problems give some idea for the way limits are used in calculus

1. The Tangent line Problem

Important Ideas:

- **Tangent line at a point**
- **Secant line**
- **Slopes**
- **Equation of a tangent line**
- **Equation of a secant line**

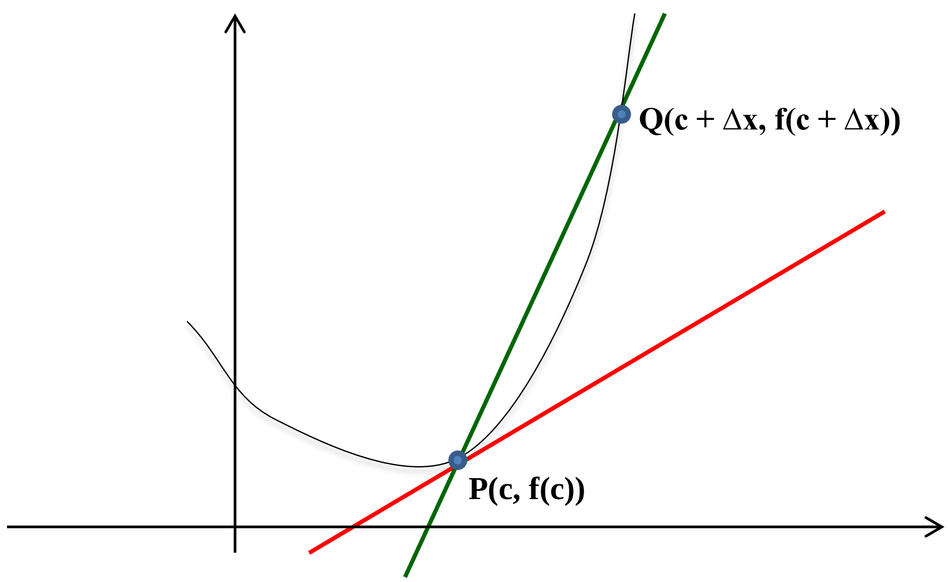
Problem: How do we find the equation of a tangent line?

Consider the following figure

Given graph of y = f(x) black line, green line (Secant Line), Red line (Tangent line).

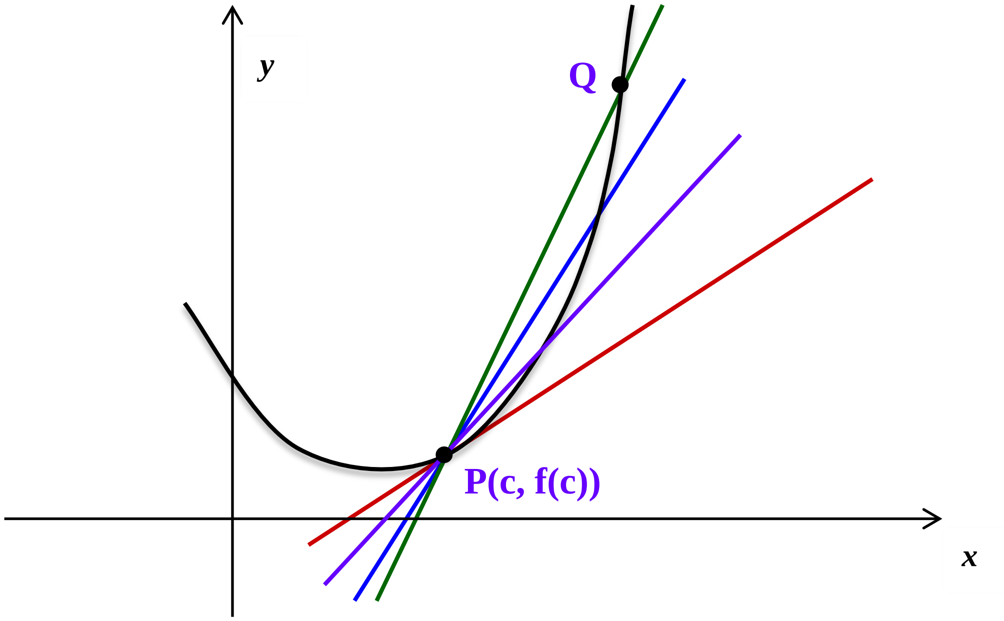
We want to find the equation of the tangent line to the curve at point P.
To fine the equation of the tangent line we need to find the slope of the line.
Q. How do you find the slope of a line?

A)



Green line, the line through the points P and Q, is a secant line to the curve $y = f(x)$

B)



As **Q** gets close to **P** the **green line** gets closer and closer to the **red line**, eventually the **green line** becomes the **red line**. This way we can find the slope of the red line.

How do we express this idea in the language of mathematics: equations and variables?

Slope of: 1) Secant lines $m_s = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(c+\Delta x) - f(c)}{c+\Delta x - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$

2) Tangent Line = Slope of the Secant line as Q gets closer and closer to P

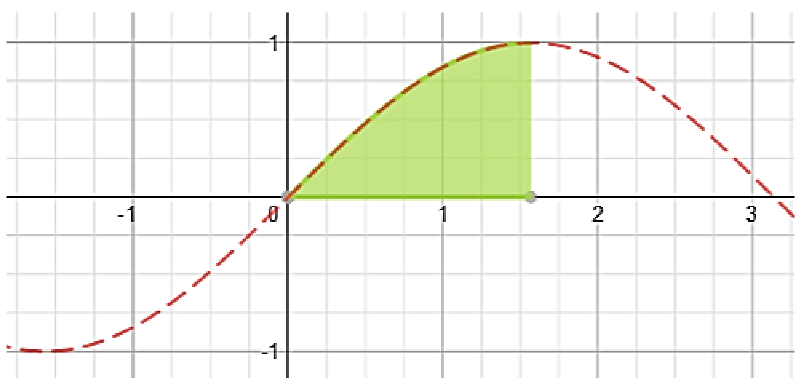
Example: Animations

- Tangent line: http://www2.latech.edu/~schroder/animations.htm
  Tangent approached by secants

2. The Area Problem

Problem Given the green region shown below we want to find the area of the region.

Area region bounded by the graph of $y = \sin x$ *on* $\left[0, \frac{\pi}{2}\right]$ and the **x-axis**



Example: Animations

- Area region bounded by the graph of $y = \sin x$ on $\left[0, \frac{\pi}{2}\right]$ and the x-axis:
  http://www2.latech.edu/~schroder/animations.htm
  left endpoints, right endpoints, midpoints

2.2 Finding Limits Graphically and Numerically (page 68)

Objectives

- Estimate a limit using a numerical or graphical approach
- Learn different ways that a limit fail to exist
- Study and use a formal definition of limit

An introduction to limits

Consider the following examples

1) Sketch the graph of the function $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$.

a) Numerically, Using Table : See below

***x* approaches 1 from the left** → ← ***x* approaches 1 from the right**

x	-1	0	0.5	0.9	0.99	0.999	…	1	…	1.0001	1.001	1.01	1.1	1.5	1.9	2
$y = f(x)$	0						…	undefined	…							3

Complete the table. Where does the value of the function $y = \frac{x^2-1}{x-1}$ gets close to when:

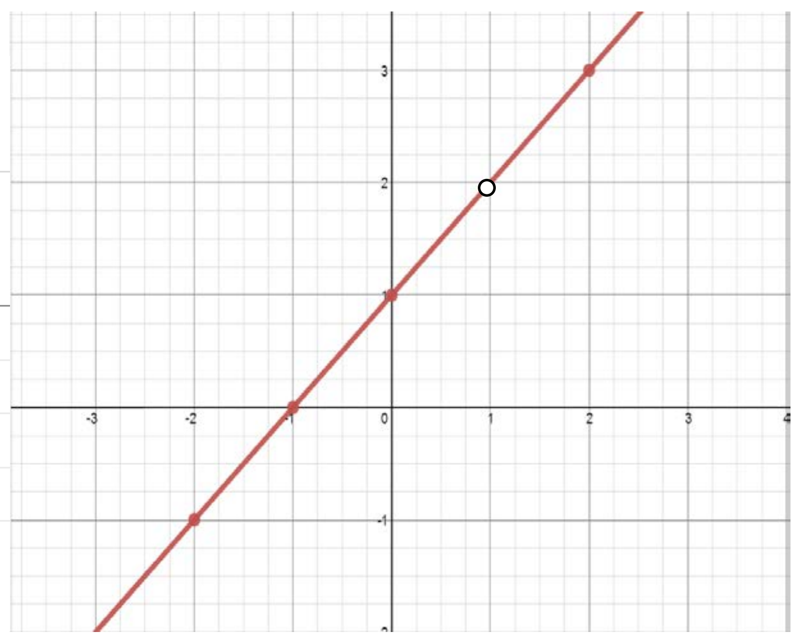
- **x gets close to 1 from the left; symbolically, as $x \longrightarrow 1^-$, $y \longrightarrow$ ______?**
- **x gets close to 1 from the right; symbolically, as $x \longrightarrow 1^+$, $y \longrightarrow$ ______ ?**

b) Graphically:

What happens to the graph as x approaches 1, both from the left and right?

$$y = \frac{(x^2-1)}{x-1}$$

x	$\frac{(x^2-1)}{x-1}$
−2	−1
−1	0
0	1
1	undefined
2	3



From the left: $If\ x \longrightarrow 1^-, then\ \ y \longrightarrow$ _____

From the right: $If\ x \longrightarrow 1^+, then\ \ y \longrightarrow$ _____

2) **Sketch the graph of** $f(x) = \frac{\sin x}{x}, x \neq 0$.

a) **Numerically, Using Table: See below**

x approaches 0 from the left →

← **x approaches 0 from the right**

x	-1	-0.5	-0.1	-0.001	-0.0001	…	0	…	0.0001	0.001	0.1	0.5	1
y						…	undefined	…					

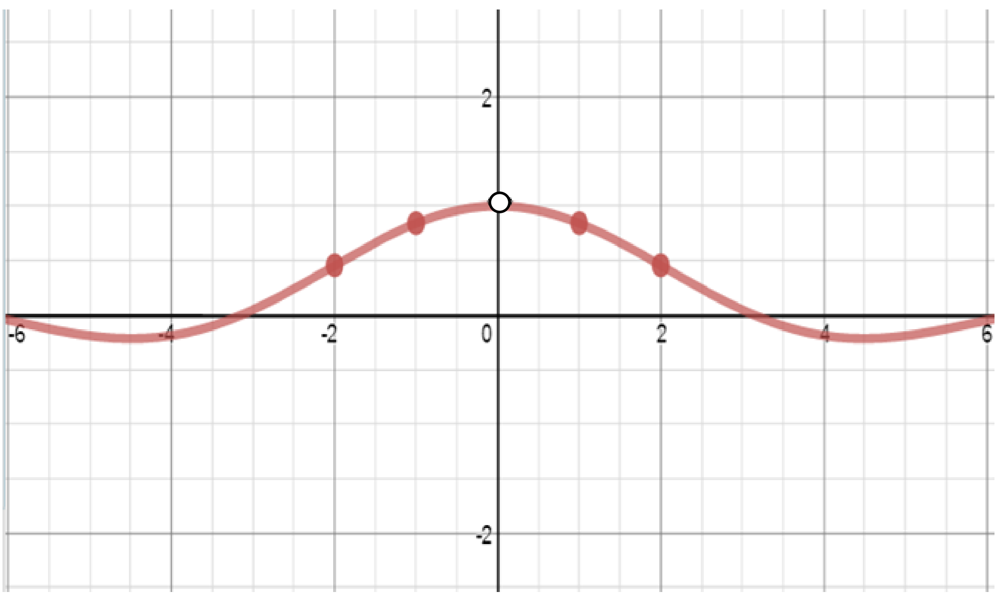
Complete the table. Where does the value of the function $y = \frac{\sin x}{x}$ **gets close to when:**

- x **gets close to 0 from the left; symbolically, as** $x \to 0^-, y \to$ ________?
- x **gets close to 0 from the right; symbolically, as** $x \to 0^+, y \to$ _______ ?

b) **Graphically**

What happens to the graph as x approaches 0, both from the left and right?

x	$\frac{\sin(x)}{x}$
−2	0.45464871
−1	0.84147098
0	undefined
1	0.84147098
2	0.45464871



From the left: $If\ x \to 1^-, then\ y \to$ ______

From the right: $If\ x \to 1^+, then\ y \to$ ______

3) **Sketch graphs or construct tables and find the following limits**

a) $\lim_{x \to -6} \frac{\sqrt{10-x} - 4}{x + 6}$ b) $\lim_{x \to 2} \frac{x/(x+1) - 2/3}{x - 2}$

Examples: YouTube Videos

- Introduction to limits 1: https://www.youtube.com/watch?v=riXcZT2ICjA
- Introduction to limits 2: https://www.youtube.com/watch?v=W0VWO4asgmk

The above examples lead to the following intuitive definition of limit

Definition (Informal):

If $\boldsymbol{f(x)}$ ***becomes*** *arbitrarily close to a single number* $\boldsymbol{L}$ *as* $\boldsymbol{x}$ ***approaches*** $\boldsymbol{c}$ ***both*** *from left and right,* *then we say the* ***limit*** *of* $\boldsymbol{f(x)}$ *as* $\boldsymbol{x}$ *approaches* $\boldsymbol{c}$ *is* $\boldsymbol{L}$*. This limit is written as:* $\boldsymbol{\lim_{x \to c} f(x) = L}$

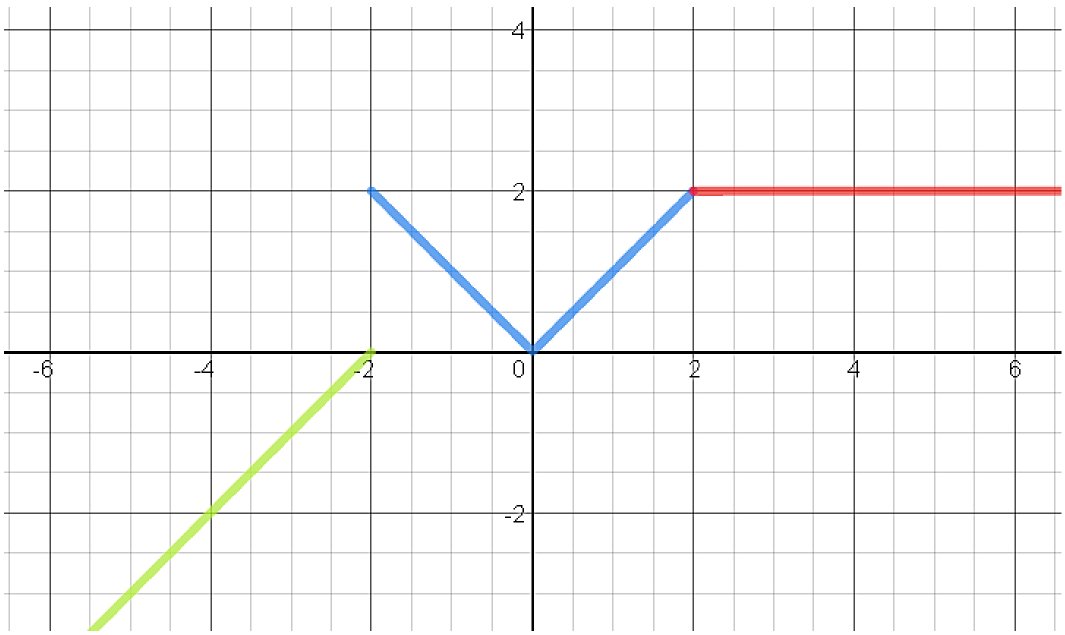
Limit Examples:

Examples: YouTube Videos

- Limit example 1: https://www.youtube.com/watch?v=GGQngIp0YGI
- Limit example 2: https://www.youtube.com/watch?v=YRw8udexH4o

Examples 1 – 2 below find the limit by referring to the graph and justify your answers

Example 1: Let $\boldsymbol{f(x) = \begin{cases} x+2, \; if\; x \le -2 \\ |x|, \; if \; -2 < x < 2 \\ 2\,, \; if \; x \ge 2 \end{cases}}$



a) $\lim_{x\to -3} f(x)$

b) $\lim_{x\to -2} f(x)$

c) $\lim_{x\to 0} f(x)$

d) $\lim_{x\to 4} f(x)$

e) $\lim_{x\to 2} f(x)$

Example 2: Let $f(x) = \begin{cases} \frac{1}{2}x - \frac{1}{2}, & if\ x \le -2 \\ 1, & if\ x > -2 \end{cases}$; Graph shown below.

Find the following limits.

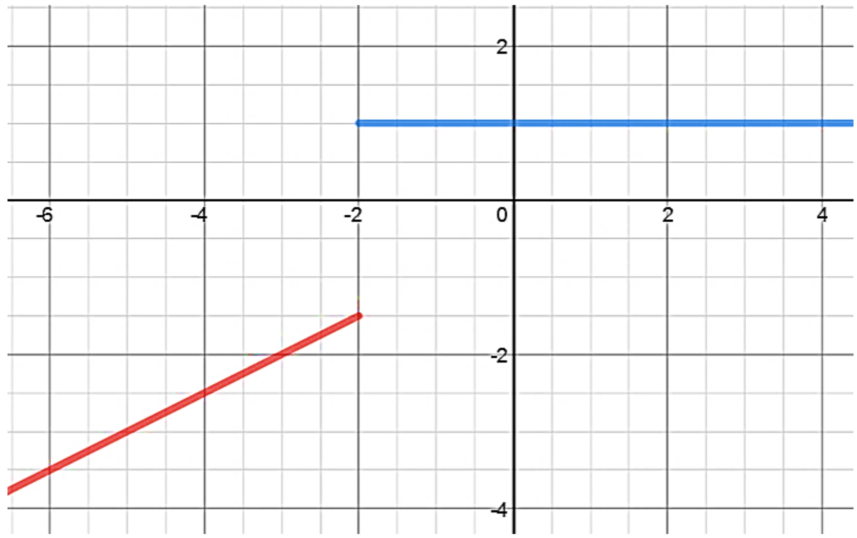
a) $\lim_{x \to -2^+} f(x) =$________

b) $\lim_{x \to -2^-} f(x) =$________

c) $\lim_{x \to -2} f(x) =$________

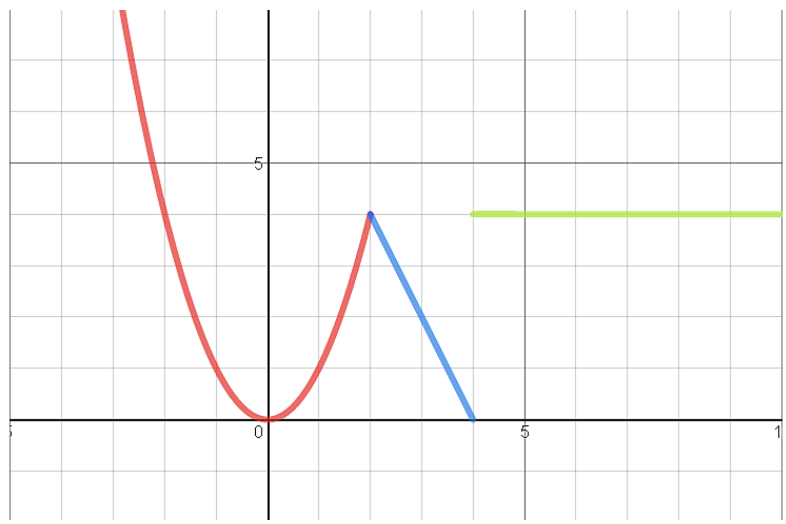
d) $\lim_{x \to 0} f(x) =$________

e) $\lim_{x \to 4} f(x) =$________

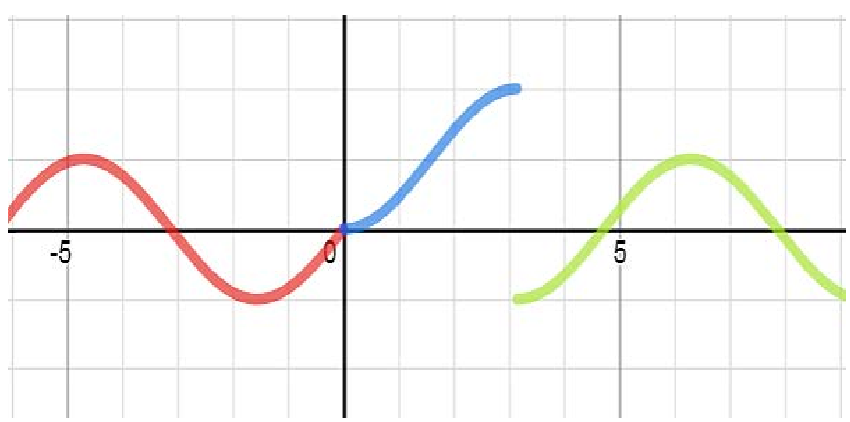


Example 3: Limits of a piecewise function. Identify the values of **c** for which $\lim_{x \to c} f(x)$ exists and explain why the limit exists at such a **c**

a) $f(x) = \begin{cases} x^2, & x \le 2 \\ 8 - 2x, & 2 < x < 4 \\ 4, & x \ge 4 \end{cases}$



b) $f(x) = \begin{cases} \sin x, & x < 0 \\ 1 - \cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$



Example 4: Sketch a graph of a function $\boldsymbol{f}$ that satisfies the given values (Several answers possible):

a) $f(0)\ is\ undefined$

$\lim_{x \to 0} f(x) = 4$

$f(2) = 6$

$\lim_{x \to 2} f(x) = 4$

b) $f(-2) = 0$

$f(2) = 0$

$\lim_{x \to -2} f(x) = 0$

$\lim_{x \to 2} f(x) = undefined$

Limits That Fail to Exist

Objectives:

- **Tell if the limit at a point exists or does not exist**
- **Explain why the limit at a point does not exist**

We proceed with examples

Examples:

1. Behavior that differs from the right and from the left. Let $\boldsymbol{f(x) = \frac{|x|}{x}}$; show that the $\lim_{x \to 0} f(x)$ does not exist.
2. Oscillating behavior: Let $\boldsymbol{f(x) = \sin\left(\frac{1}{x}\right)}$; show that the $\lim_{x \to 0} f(x)$ does not exist.
3. Unbounded behavior: Let $\boldsymbol{f(x) = \frac{1}{x^2}}$, show that the $\lim_{x \to 0} f(x)$ does not exist.

One Sided Limits / Two Sided Limits

Objectives:

- **Identify left hand limits and right hand limits**
- **Find the limit of a function from the left and right**
- **Find one sided limit**
- **Explain the relationships between one sided limit and two sided limit**
- **Discuss existence or non- existence of a limit in terms of One Sided Limits**

One Sided Limits:

Limit from the left at c is denoted by

$\lim_{x \to c^-} f(x)$ Here x **approaches** c from the **left**

Limit from the right at c is denoted by

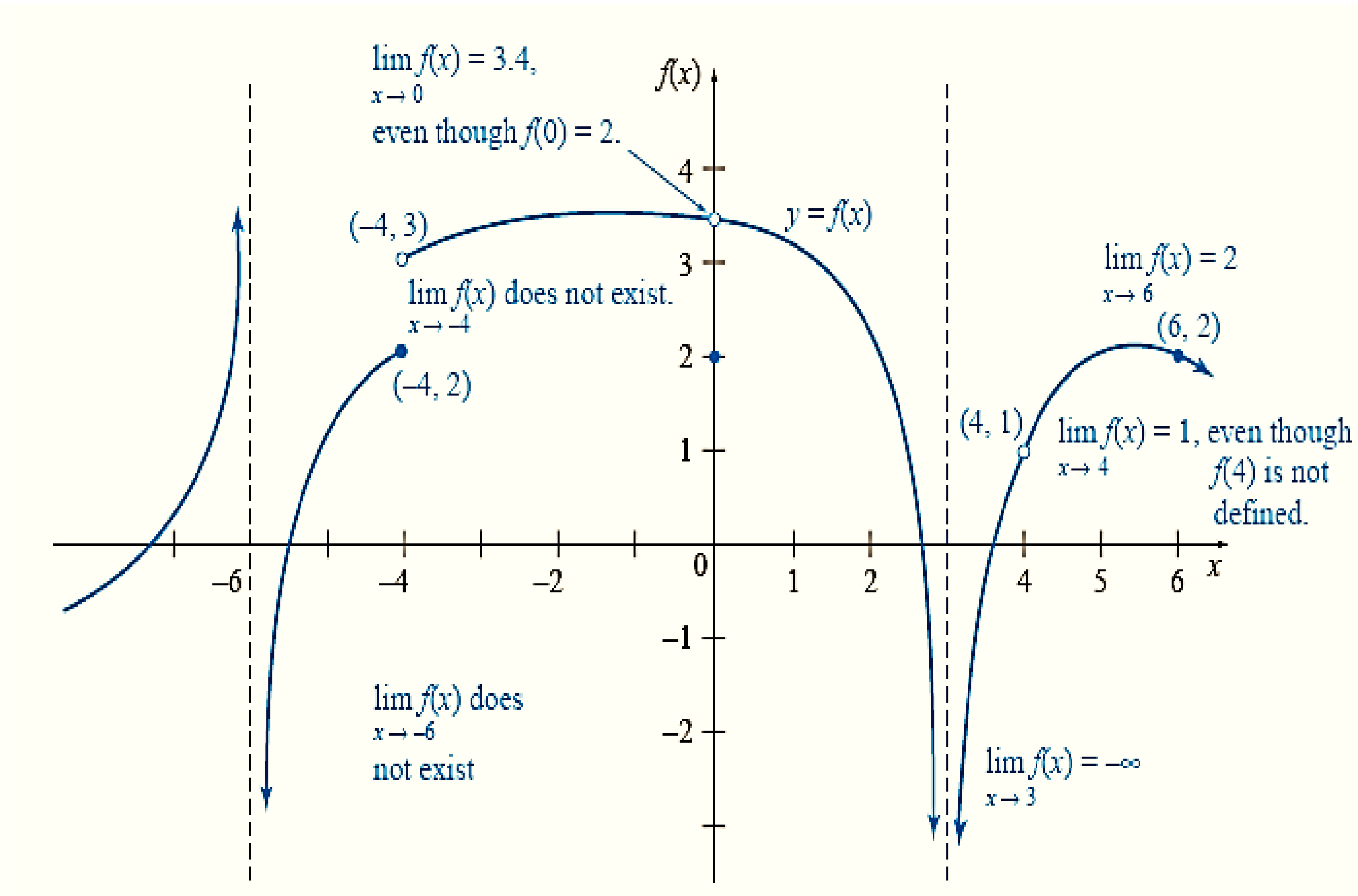
$\lim_{x \to c^+} f(x)$ Here x **approaches c** from the **right**

Two Sided Limits:

$\lim_{x \to c} f(x)$ Limit as x ***approaches*** c **both** from the **left and right**

Note: $\lim_{x \to c} f(x)$ exists **if and only if** both **right hand and left hand** limits at **c exist** and are **equal**

Example 1: Let's look at a graph to see what types of limits we get.



Example: Online Videos Khan Academy lectures and exercises

- One sided limits from graphs: https://www.khanacademy.org/math/differential-calculus/limits_topic/calculus-estimating-limits-graph/v/one-sided-limits-from-graphs
- Two sided limits from graphs: https://www.khanacademy.org/math/differential-calculus/limits_topic/calculus-estimating-limits-graph/v/2-sided-limit-from-graph
- Determine which limits are true: https://www.khanacademy.org/math/differential-calculus/limits_topic/calculus-estimating-limits-graph/v/determining-which-limit-statements-are-true

Example 2: For each of the following find the indicated limit, if exists, by referring to the graph given on the right hand side and justify your answer

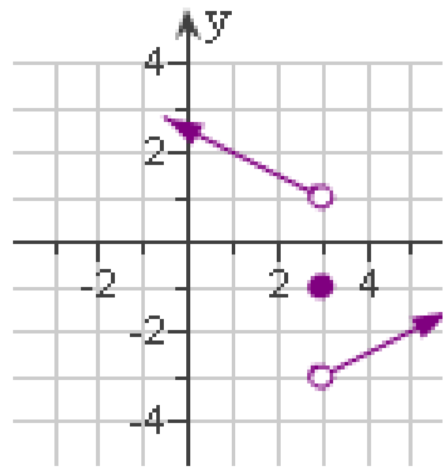
i.

a) $\lim_{x \to 3^-} f(x) =$

b) $\lim_{x \to 3^+} f(x) =$

c) $\lim_{x \to 3} f(x) =$

d) What is $f(2) =$_______?



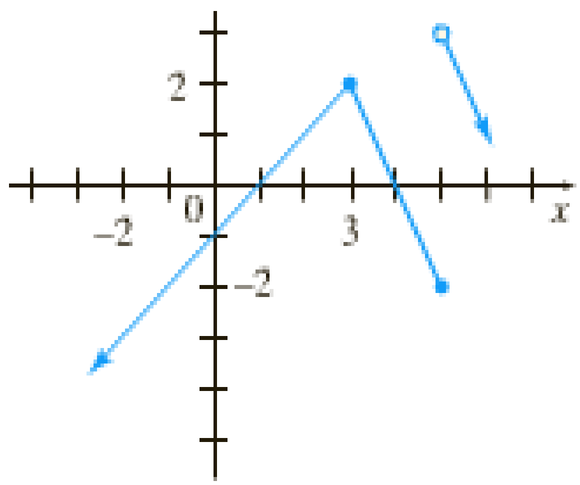
ii. **a)** $\lim_{x \to 3} f(x) =$

b) $\lim_{x \to 5^+} f(x) =$

c) $\lim_{x \to 4} f(x) =$

d) $\lim_{x \to 5} f(x) =$

e) $\lim_{x \to 0} f(x) =$

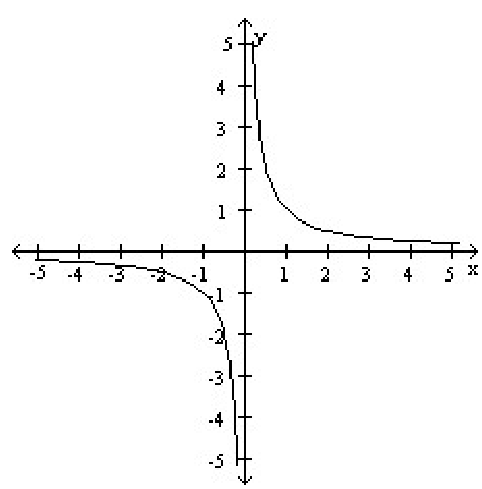


iii. **a)** $\lim_{x \to 0^+} f(x) =$

b) $\lim_{x \to 0^-} f(x) =$

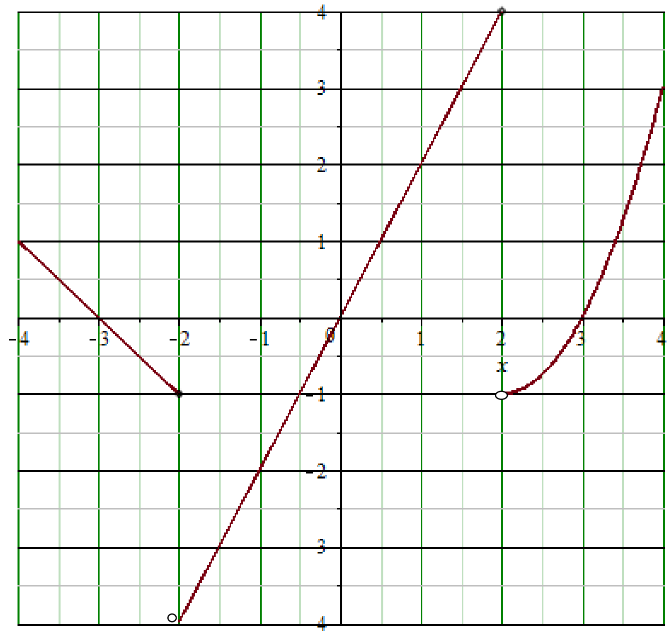
c) $\lim_{x \to 3} f(x) =$

d) $\lim_{x \to -2} f(x) =$



Example 3: Let $f(x) = \begin{cases} -3-x, & if\ x \le -2 \\ 2x\ , & if -2 < x \le 2 \\ x^2-4x+3\ , & if\ x > 2 \end{cases}$, graph shown below

Find the following limits:



a) $\lim_{x \to -2^+} f(x) =$

b) $\lim_{x \to -2^-} f(x) =$

c) $\lim_{x \to 2^+} f(x) =$

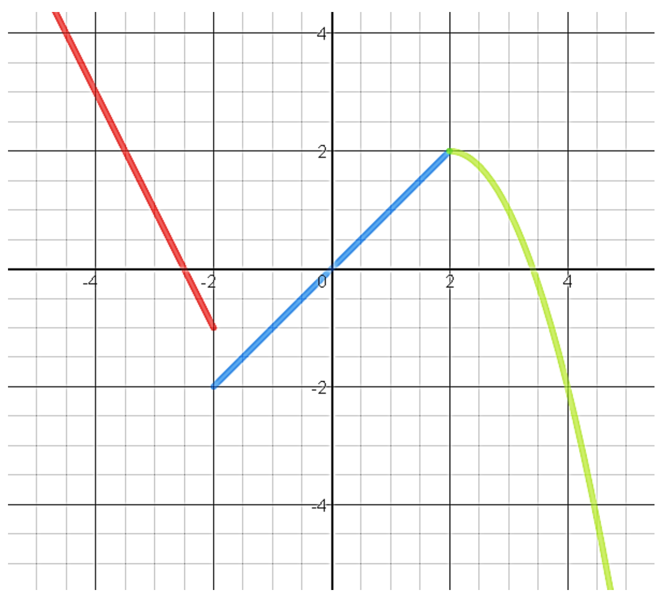
d) $\lim_{x \to 2^-} f(x) =$

e) $\lim_{x \to 2} f(x) =$

f) $\lim_{x \to 0} f(x) =$

Example 3: Let $f(x) = \begin{cases} -4-x, & if\ x < -2 \\ x\ , & if -2 \le x < 2 \\ -x^2+4x-2\ , & if\ x \ge 2 \end{cases}$ See graph below

Find the following limits:



a) $\lim_{x \to 2^+} f(x) =$

b) $\lim_{x \to 2^-} f(x) =$

c) $\lim_{x \to 2} f(x) =$

d) $\lim_{x \to -2} f(x) =$

e) $\lim_{x \to -2^+} f(x) =$

f) $\lim_{x \to -2^-} f(x) =$

The Formal Definition of Limit (page 72)

Objectives:

- **State the formal Definition of Limit**
- **Understand/Explain the formal definition**
- **Use the formal definition to prove/find the limit of a function if it exists**

Recall the intuitive definition of limit: $\lim_{x \to c} f(x) = L$

If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from left and right, then we say the limit of $f(x)$ as x approaches c is L.

This definition lacks the mathematical rigor and prerecession.

For example: How close; is gets close mean?

We need to have a measure for getting close

Definition (Formal or the $\varepsilon - \delta$ Definition):

Let f be a function defined on an open interval containing c (except possibly at c), and L be a real number. The statement: " $\lim_{x \to c} f(x) = L$" means, **for every $\varepsilon > 0$ there exists a $\delta > 0$,** such that:

If $0 < |x - c| < \delta$, **then** $|f(x) - L| < \varepsilon$

Examples YouTube video

- Epsilon-delta limit definition 1: https://www.youtube.com/watch?v=-ejyeII0i5c
- Epsilon-delta definition of limits: https://www.youtube.com/watch?v=w70af5Ou70M
- Epsilon-delta limit definition 2: https://www.youtube.com/watch?v=Fdu5-aNJTzU
- Proving a Limit: x^2 = 4: https://www.youtube.com/watch?v=gLpQgWWXgMM

Examples:

a) Let $f(x) = L$ and **c** be any real number; using the formal definition of limit show that $\lim_{x \to c} f(x) = L$

b) Using the formal definition of limit show that $\lim_{x \to c} x = c$

c) Using the formal definition of limit show that $\lim_{x \to 3} (2x - 5) = 1$

d) Using the formal definition of limit show that $\lim_{x \to 3} (2x + 2) = 8$

e) Using the formal definition of limit show that $\lim_{x \to 2} x^2 = 4$

Practice Problems:

Larson Book 5th ed. Exercise 2.2 Page 75-77, 9 – 33 odd numbered problems, 41, 43, 45, 51, 53, 55

Page 76 question # 35 – 38. Find the Limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

35) $\lim_{x\to 2}(3x + 2)$ 36) $\lim_{x\to 6}\left(6 - \frac{x}{3}\right)$ 37) $\lim_{x\to 2}(x^2 - 3)$ 38) $\lim_{x\to 4}(x^2 + 6)$

2.3 Evaluating Limits Analytically (page 79)

Objectives: By the end of this section you should be able to:

- Evaluate limits using properties of limits
- Develop and use strategy for finding limits
- Evaluate limit using the Squeeze Theorem.

Properties of Limits

Theorem 2.1 Some Basic Limits

Let b *and* c be real numbers and n be a positive integer; then

1) $\lim_{x \to c} b = b$
2) $\lim_{x \to c} x = c$
3) $\lim_{x \to c} x^n = c^n$

The proof Theorem 2.1 can easily follow from the Formal Definition of limit

Theorem 2.2 Properties of Limit

Let b and c be real numbers and n be a positive integer, and let f and g be functions with the following limits: $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$, then

1) $\lim_{x \to c} bf(x) = bL$ (Scalar Multiple Rule)
2) $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$ (Sum/Difference Rule)
3) $\lim_{x \to c} [f(x)g(x)] = LK$ (Product Rule)
4) $\lim_{x \to c} [f(x)/g(x)] = L/K,\ K \neq 0$ (Quotient Rule)
5) $\lim_{x \to c} [f(x)]^n = L^n$ (Power Rule)

Examples: Given $\lim_{x \to 4} f(x) = 16$ and $\lim_{x \to 4} g(x) = 8$ find:

a) $\lim_{x \to 4} [f(x) - g(x)] =$

b) $\lim_{x \to 4} [f(x)g(x)] =$

c) $\lim_{x \to 4} \frac{f(x)}{g(x)} =$

d) $\lim_{x \to 4} [f(x) + 5g(x)] =$

e) $\lim_{x \to 4} [f(x)]^3 =$

Theorem 2.3 Limits of Polynomial and Rational Functions

1) If $\boldsymbol{p}$ is a polynomial function and $\boldsymbol{c}$ is a real number, then $\lim_{x \to c} p(x) = p(c)$
2) If $\boldsymbol{r}$ is a rational function given by $r(x) = p(x)/q(x)$ and $\boldsymbol{c}$ is a real number such that $q(c) \neq 0$, then $\lim_{x \to c} r(x) = r(c) = p(c)/q(c)$.

Examples: Find the limits of each of the following.

a) $\lim_{x \to 2} (x^2 + 2x - 3)$

b) $\lim_{x \to -6} \frac{\sqrt{10-x} - 4}{x + 6}$

c) $\lim_{x \to 2} \left(\frac{x^2-7x+10}{x+2}\right)$

d) $\lim_{x \to 2} \frac{x/(x+1) - 2/3}{x - 2}$

Example: For each of the following find the limit.

a) $\lim_{x \to 2} \frac{|x+3|}{5}$

b) $\lim_{x \to 1} \left(\frac{x^2-1}{x-1}\right)$

c) $\lim_{x \to 2} \left(\frac{x^2-7x+10}{x-2}\right)$

d) $\lim_{x \to 0} \left(\frac{\frac{1}{x+4} - \frac{1}{4}}{x}\right)$

Examples: Example 2 & 3 page 80

Theorem 2.4: The Limit of a Function Involving Radicals

Let $\boldsymbol{n}$ be a **positive integer; then:** The following Limits are valid.

g) $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ for $c > 0$ **if n is even**

h) $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ **for all c if n is odd** and

Theorem 2.5: The Limit of a Composite Function

If f *and* g are functions such that: $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f(L) = f\left(\lim_{x \to c} g(x)\right)$$

Example: Example 4 page 81

Example: Find $\lim_{x \to 1} \sqrt[3]{x^4 - 4x + 28}$

Examples YouTube video

- limit examples: https://www.youtube.com/watch?v=gWSDDopD9sk
- limit examples: https://www.youtube.com/watch?v=xjkSE9cPqzo
- Limit examples:

Practice Problems

Wikibook; use e-Book link: Exercise section 2.7: Basic limit exercises #1 – 26, 37 – 39

Theorem 2.6 Limits of Transcendental Functions

Let c be a real number in the domain of the given trigonometric function. Then

1) $\lim_{x \to c} \sin x = \sin c$
2) $\lim_{x \to c} \cos x = \cos c$
3) $\lim_{x \to c} \tan x = \tan c$
4) $\lim_{x \to c} a^x = a^c\ ,\ a > 0$
5) $\lim_{x \to c} \csc x = \csc c$
6) $\lim_{x \to c} \sec x = \sec c$
7) $\lim_{x \to c} \cot x = \cot c$
8) $\lim_{x \to c} \ln x = \ln c$

Examples: Find the following limits

a) $\lim_{x \to \pi/3} 3 \sin x =$

b) $\lim_{x \to 1} (2x - 3 \ln x + 4) =$

c) $\lim_{x \to 1} \left(x2^x + 2 \ln x - 2\sqrt[3]{9 - x} + \tan\left(\frac{\pi}{4}x\right)\right) =$

d) $\lim_{x \to \pi/3} (3 \sin 2x + \cos 6x - \cot x) =$

e) $\lim_{x \to e} (xe^{e-x} + 2 \ln x) =$

f) $\lim_{x \to 3} \left(\frac{1}{3}x^2 - \frac{2}{3}x + \csc\left(\frac{1}{2}\pi x\right) - \frac{5}{2}\right) =$

A strategy for Finding Limits

Here we learn how to find limits using

- **Theorem 2.7**
- **Dividing Out and Rationalizing Techniques**
- **The Squeeze Theorem**

Theorem 2.7 Functions That Agree at All but One Point

Let c be any real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c.

If the limit of $g(x)$ **as** x **approaches** c **exists**, then the limit of $f(x)$ **as** x **approaches** c also **exists** and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

Example: $f(x) = \frac{x^2 - 4}{x - 2}$ **and** $g(x) = x - 2$, $f(x)$ and $g(x)$ agree everywhere except at $x = -2$.
Thus by **Theorem 2.7** they have the same limit at $x = -2$. **That is** $\lim_{x \to -2} f(x) = \lim_{x \to -2} g(x) = -4$

Example: Example 6 page 82

Dividing Out and Rationalizing Techniques

Examples: Find the following Limits

a) $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$

b) $\lim_{x \to 36} \left(\frac{\sqrt{x} - 6}{x - 36}\right)$

c) $\lim_{x \to 2} \frac{x^2 - 7x + 10}{x - 2}$

d) $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$

Example: Example 7 page 83

Example: Example 8 page 84

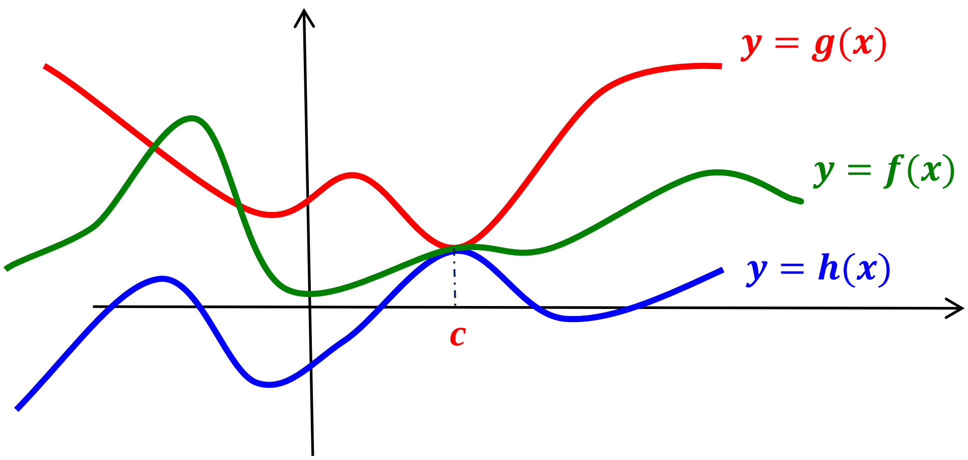
The Squeeze Theorem (page 85)

Theorem 2.8: The Squeeze Theorem (sandwich theorem)

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c,

and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$, **then** $\lim_{x \to c} f(x) = L$

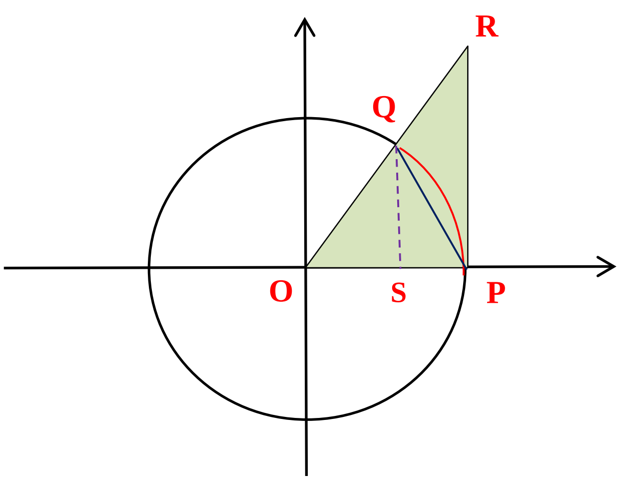
Pictorially



Theorem 2.9 Three Special Limits

1) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ 2) $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$ 3) $\lim_{x\to 0}(1+x)^{1/x} = e$

Proof of 1) Consider the **unit circle** and the two different **triangles** shown in the figure



Let measure of angle ROP $= x$

We have $\Delta\, OPR$ **is bigger** *than* ΔOQP and

ΔOQP *is* **smaller than** *sector OPQ*

The area of the different regions is given by

$a(\Delta\, OPR) = \frac{1}{2}\ OP * PR = \frac{1}{2}\ \tan x$

$a(\text{sector } OPQ) = \frac{x}{2}$, *and*

$a(\Delta\, OPQ) = \frac{1}{2} OP * SQ = \frac{1}{2}\ \sin x$

From the figure we see that $a(\Delta\, OPQ) \leq a(\text{sector } OPQ) \leq a(\Delta\, OPR)$, that is

$\frac{1}{2}\ \sin x \ \leq \frac{x}{2} \leq \frac{1}{2}\ \tan x$, which gives

$\cos x \ \leq \frac{\sin x}{x} \leq 1$, then

Squeeze Theorem gives $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Example YouTube Video

- Squeeze theorem (For (1) above): https://www.youtube.com/watch?v=igJdDN-DPgA

Limit Involving Trigonometric Functions

Examples: Example 9 & 10 page 86

Example: Find the following limits

a) $\lim_{x \to 0} \frac{\tan x}{x}$

b) $\lim_{x \to 0} \frac{1 - \cos x}{x}$

Example: Show that $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$

Exercises 2.3 Page 87: 3, 11, 15, 21, 25, 31, 35, 39, 47, 49, 53, 59, 65, 67, 69, 71, 77, 80, 91, 92, 93

Practice Problems:

Review Worksheet 1 on Limits

2.4 Continuity and One Sided Limits (page 90)

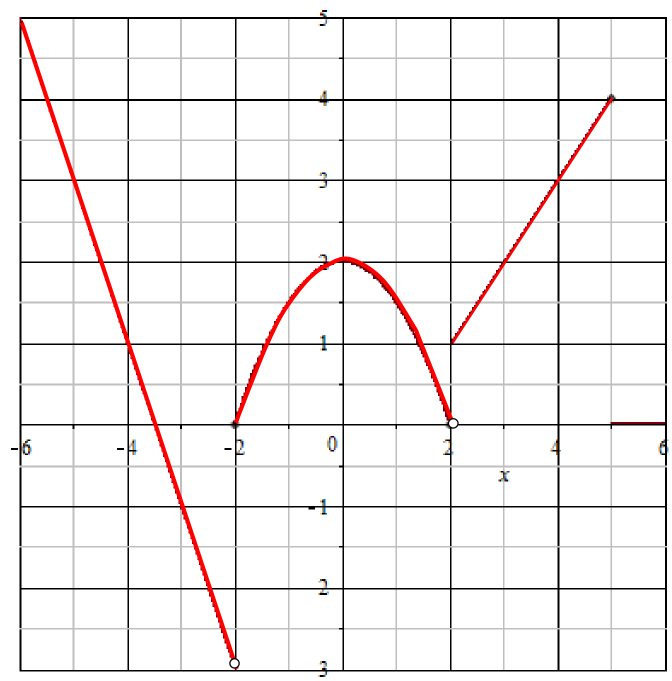
Note: We already discussed One Sided Limits

Objectives: By the end of this section you should be able to

- **Define Continuity at a point**
- **Determine continuity at a point and continuity on an open interval**
- **Determine one sided limits and continuity on a closed interval**
- **Use properties of continuity**
- **Understand the Intermediate Value Theorem**

One sided Limits

Example: $f(x) = \begin{cases} -2x - 7, & if \quad x < -2 \\ -\frac{1}{2}x^2 + 2, & if \ -2 \le x < 2 \\ x - 1, & if \ \ 2 < x \le 5 \end{cases}$ Find:



a) $\lim_{x \to -2^-} f(x)$

b) $\lim_{x \to -2^+} f(x)$

c) $\lim_{x \to -2} f(x)$

d) $\lim_{x \to -3} f(x)$

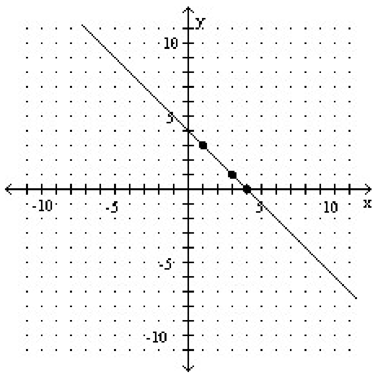
e) $\lim_{x \to 2^+} f(x)$

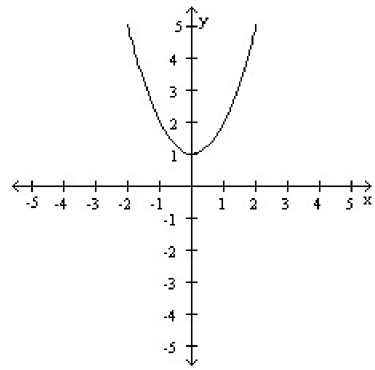
f) $\lim_{x \to 0^-} f(x)$

Continuity

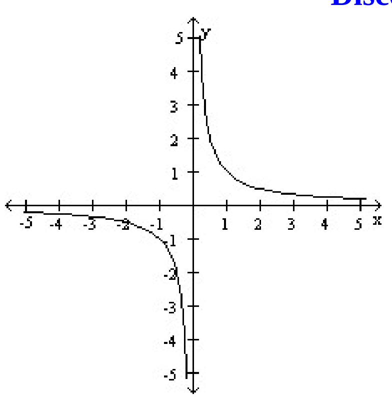
Continuous: **No Break, No Jump, and No Asymptotic Approach**
Consider the following graphs

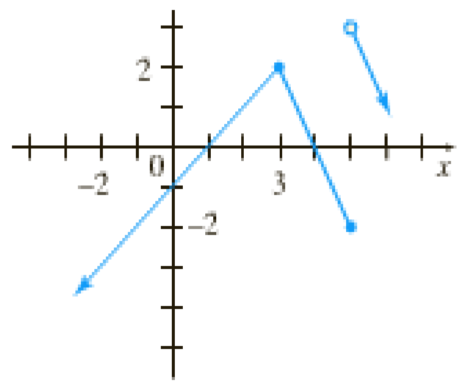
Continuous Graphs





Discontinuous Graphs





To be continuous – you should be able to draw the graph completely without lifting your pencil

Definition: Continuity

a. A function f is said to be continuous at $x = c$ if and only if the following **three** conditions are satisfied**:**

1) $f(c)$ is **defined**

2) $\lim_{x \to c} f(x)$ **exists**

3) $\lim_{x \to c} f(x) = f(c)$

b. A function f is continuous **on an open interval** (a, b) if it is **continuous** at **each point** of the open **interval. A** function that is continuous on the **entire real line** $(-\infty, \infty)$ is **everywhere continuous**.

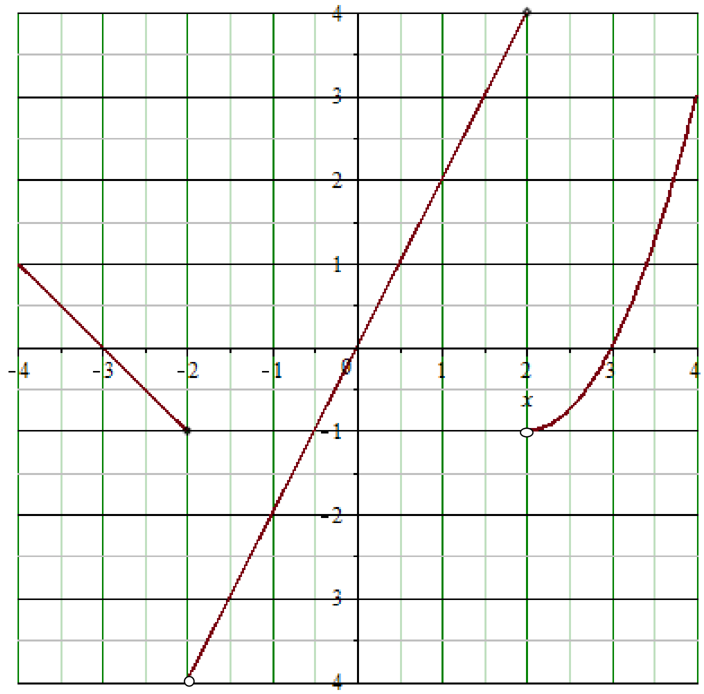
Example: YouTube videos

- Limits to define continuity: https://www.youtube.com/watch?v=kdEQGfeC0SE

Example: Example 1 page 91

Example 1: Discuss the continuity of the function

$$f(x) = \begin{cases} -3 - x, & if \quad x \le 2 \\ 2x, & if \quad -2 < x \le 2 \\ x^2 - 4x + 3, & if \quad x > 2 \end{cases}$$, graph shown below



Example 2: Discuss the **continuity** of each function

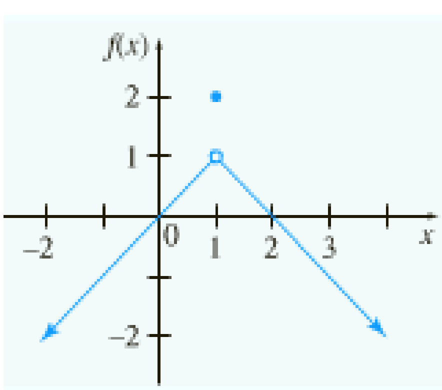
a) $f(x) = \frac{x^2 + x - 6}{x + 3}$

b) $f(x) = \frac{x^2 - 7x + 10}{x - 2}$

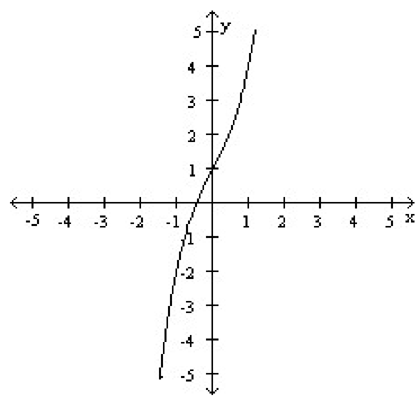
c) $f(x) = \frac{\sqrt{4 - x^2}}{x}$

Example 3: referring to the graphs find all points where the function is **discontinuous.**

a)



b)



Example 4: Find all values of $x = a$ where the function is **discontinuous**.

a) $f(x) = \frac{x+3}{x(x+2)}$

b) $f(x) = \frac{x^2-9}{x+3}$

c) $f(x) = x^2 + 3x + 1$

d) $f(x) = \frac{|x+2|}{x+2}$

Example: YouTube video

- Limit at a point of discontinuity: https://www.youtube.com/watch?v=Y7sqB1e4RBI

Example: Example 2&3 page 92

Definition (Continuity on a Closed Interval)

A function f is continuous on the closed interval $[a, b]$ if it is **continuous** on the **open** interval (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ **and** $\lim_{x \to b^-} f(x) = f(b)$.

The function f is **continuous** from the **right at** a and from the **left at b**

Example: Example 4 page 93

Example 1: Continuity on a closed interval. Discuss the continuity of $f(x) = \sqrt{x^2 - 4}$

Example 2: Describe the interval for which the function $f(x) = csc\ x$ is continuous.

Properties of Continuity

Theorem 2.11: Let b be a real number. If f *and* g are continuous at $x = c$, then the following functions are also continuous at c.

a) Scalar multiple bf

b) Product fg

c) The sum or difference $f \pm g$

d) Quotient f/g, provided $g(c) \neq 0$

Lists of familiar functions that are continuous in their domain

1) Polynomial
2) Rational
3) Trigonometric
4) Exponential and logarithmic

Example: Example 6 page 94

Theorem 2.12: Continuity of a composite function

If g is continuous at c and f is continuous at $g(c)$, then the **composite function** given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Proof: By the definition of continuity: $\lim_{x \to c} g(x) = g(c)$ **and** $\lim_{x \to g(c)} f(g(x)) = f(g(c))$

By limit **Theorem on Composite functions**

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(g(c)) \quad \textbf{QED}$$

Example: Example 7 page 96

Example 3: Describe the **interval** in which the **function** $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ is **continuous**

Example: YouTube videos

- Finding a limit to function continuous: https://www.youtube.com/watch?v=P1DJxuG7U9A

Example 4: For the piecewise defined functions below, describe the interval in which each function is continuous.

a) $f(x) = \frac{x+3}{x^2-4x+4}$

b) $f(x) = \begin{cases} 1, & if\ x < 2 \\ x+3, & if\ 2 \leq x \leq 4 \\ 7, & if\ x > 4 \end{cases}$

c) $h(x) = \begin{cases} 4x+4, & if\ x \leq 0 \\ x^2 - 4x + 4, & if\ x > 0 \end{cases}$

d) $g(x) = \begin{cases} x^2+1, & if\ x < -2 \\ 2x+1, & if\ -2 \leq x \leq 3 \\ 4, & if\ 3 < x < 5 \\ x-1, & if\ x \geq 5 \end{cases}$

Theorem 2.13: Intermediate value Theorem

If $\boldsymbol{f}$ is **continuous** on a closed interval $[\boldsymbol{a},\ \boldsymbol{b}]$, *and* $\boldsymbol{f}(\boldsymbol{a}) \neq \boldsymbol{f}(\boldsymbol{b})$**,** and $\boldsymbol{k}$ is any number between $\boldsymbol{f}(\boldsymbol{a})$ and $\boldsymbol{f}(\boldsymbol{b})$**,** then **there** is **at least one number** $\boldsymbol{c}$ in $[\boldsymbol{a}, \boldsymbol{b}]$ such that $\boldsymbol{f}(\boldsymbol{c}) = \boldsymbol{k}$**.**

Example 5: Use the **intermediate value theorem** to show that the function $\boldsymbol{f(x) = x^3 - 3x + 1}$ has at least one zero in the interval $[\boldsymbol{-2, -1}]$.

Solution:

- First notice that $\boldsymbol{f(x) = x^3 - 3x + 1}$ is continuous on $[\boldsymbol{-2,-1}]$ **and** $\boldsymbol{f(-2) \neq f(-1)}$
- Check the values of the function at the end points

  $\boldsymbol{f(-2) = (-2)^3 - 3(-2) + 1 = -1 < 0}$

  $\boldsymbol{f(-1) = (-1)^3 - 3(-1) + 1 = 3 > 0}$

Note that, $f(-2) = -1$and $f(-1) = 3$ have opposite signs. This implies that$f(-2) < 0 < f(-1)$; that is, 0 is between $f(-2)$and $f(-1)$.

- Thus by The Intermediate Value Theorem there is at least one c in $(\boldsymbol{-2,-1})$ such that $\boldsymbol{f(c) = 0}$**.** Which is the same as saying $\boldsymbol{f}$ has at least one zero in the interval $(\boldsymbol{-2,-1})$.

Example 6: Use the intermediate value theorem to show that for all **spheres** with **radii** in the interval [**5**, **8**] there is one with a volume of **1500** cubic centimeters.

Example 7: Prove that if $\boldsymbol{f}$ is continuous and have **no zeros** on $[\boldsymbol{a},\ \boldsymbol{b}]$, then either $\boldsymbol{f(x)} > \boldsymbol{0}$ for all $\boldsymbol{x}$ *in* $[\boldsymbol{a},\ \boldsymbol{b}]$ ***or*** $\boldsymbol{f(x)} < 0$ for all $\boldsymbol{x}$ ***in*** $[\boldsymbol{a},\ \boldsymbol{b}]$.

Practice Problems:

Page 98 Exercises 2.4 ; 1 – 6, 9, 11, 15, 16, 17, 23, 25, 31, 33, 36, 43, 47, 53, 57, 59, 61, 69, 71, 72, 76, 84

Practice Problems:
Review Worksheet 1 on Limits

2.5 Infinite Limits (Page 103)

Objectives:

- Understand Infinite Limits
- Determine infinite limits from the right and from the left
- Find the vertical asymptote of a graph
- Understand the formal definition of infinite limits

Example 1: Consider the function f(x) = 1/x , sketch the graph and find

a) $\lim_{x \to 0^-} f(x)$ The limit as x approaches 0 from the left

x approaches 0 from the left →

x	**-1**	**-0.5**	**-0.1**	**-0.001**	**-0.0001**	**…**	**0**
y						**…**	

b) $\lim_{x \to 0^+} f(x)$ The limit as x approaches 0 from the right

← **x approaches 0 from the right**

x	**0**	**…**	**0.0001**	**0.001**	**0.1**	**0.5**	**1**
y		**…**					

Example 2: Consider the function $f(x) = 1/x^2$, sketch the graph and find

a) $\lim_{x \to 0^-} f(x)$

b) $\lim_{x \to 0^+} f(x)$

c) $\lim_{x \to 0} f(x)$

What is the difference between **Example 1, Example 2?**

Note: Infinite limits, "functional values" are $\pm\infty$

Definition of Infinite Limits

Let f be a function that is defined at every number in an open interval containing the number c except possibly at c itself. The statement

a) $\lim_{x \to c} f(x) = \infty$

Means that for each M > 0 there is a δ > 0 such that 0 < | x – c | < δ implies that f(x) > M

b) $\lim_{x \to c} f(x) = -\infty$

Means that for each N < 0 there is a δ > 0 such that 0 < | x – c | < δ implies that f(x) < N

Definition of Vertical Asymptote

If $\boldsymbol{f(x)}$ approaches to infinity (or minus infinity) as $\boldsymbol{x}$ approaches $\mathbf{0}$ from the right or from the left, then the line $\boldsymbol{x = c}$ is the vertical asymptote for the graph of $\boldsymbol{f}$.

Theorem 2.14 Vertical Asymptotes

Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be continuous on an open interval containing *c*. If $\boldsymbol{f(c) \neq 0}$ and $\boldsymbol{g(c) = 0}$, and there exists an **open** interval **containing** $\boldsymbol{c}$, such that $\boldsymbol{g(c) \neq 0}$ for all $\boldsymbol{x \neq c}$ in the interval, then the graph of the function given by $\boldsymbol{h(x) = \frac{f(x)}{g(x)}}$ has a vertical asymptote at $\boldsymbol{x = c}$ **.**

Example 3: Find vertical asymptotes.

a) $\boldsymbol{f(x) = \frac{1}{x+2}}$

b) $\boldsymbol{f(x) = \frac{x-1}{x}}$

c) $\boldsymbol{f(x) = \frac{x^2+1}{x^2-4}}$

d) $\boldsymbol{f(x) = \frac{x^2-1}{x(x+1)}}$

Practice Problems
Exercises 2.5, Page 108 - : 1 – 8, 13, 15, 17, 19, 27, 29, 41

Practice Problems:
Review Worksheet 1 on Limits

Limits at Infinity and Horizontal Asymptotes (Page 238)

In this section we consider limit of the types:

1) $\lim_{x \to \infty} f(x)$; that is x is **increasing** without bound

2) $\lim_{x \to -\infty} f(x)$; that is x is **decreasing** without bound

Example 1: Find the following limits

a) $\lim_{x \to \infty} \left(\frac{1}{x}\right)$

b) $\lim_{x \to -\infty} \left(\frac{1}{x}\right)$

c) $\lim_{x \to \infty} \left(\frac{1}{x^2}\right)$

Note: In general, for any positive real number n

d) $\lim_{x \to \infty} \left(\frac{1}{x^n}\right) = 0$ and $\lim_{x \to -\infty} \left(\frac{1}{x^n}\right) = 0$

To evaluate limits at infinity, we can divide the rational function (both numerator and denominator) by the largest power of the variable that appears in the denominator.

Consider the following Example;

Example: Find the limit $\lim_{x \to \infty} \frac{x^2 + 2x - 5}{3x^2 + 2}$

Solution:

$$\lim_{x \to \infty} \frac{x^2 + 2x - 5}{3x^2 + 2} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + \frac{2x}{x^2} - \frac{5}{x}}{\frac{3x^2}{x^2} + \frac{2}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{2}{x} - \frac{5}{x}}{3 + \frac{2}{x^2}}$$

$$= \frac{1 + 0 - 0}{3 + 0}$$

$= 1/3$, (take the limit of each term in the numerator and denominator)

Note: The line $y = 1/3$ is the Horizontal Asymptote for the function $f(x) = \frac{x^2 + 2x - 5}{3x^2 + 2}$

Definition: (Horizontal Asymptote)

The Line $\boldsymbol{y = L}$ is a Horizontal Asymptote of the graph of $\boldsymbol{f}$ if $\lim_{x \to \pm\infty} f(x) = L$

Example 2:

a) $\lim_{x \to \infty} \left(\frac{1}{x}\right) = \mathbf{0}$; implies that the line $\boldsymbol{y = 0}$ is the horizontal asymptote for the graph of $\boldsymbol{f(x) = \frac{1}{x}}$.

b) Find the horizontal asymptote for $f(x) = \frac{-3x^3 + 2x^2 + 15}{2x^3 + 2x - 11}$

Example 3: Find the limit and the Horizontal Asymptote

a) $\lim_{x \to \infty} \frac{2x^3 + 3x - 1}{3x^3 - 2}$

b) $\lim_{x \to \infty} \frac{3x^3 + 2x - 1}{2x^4 - 3x^3 - 2}$

c) $\lim_{x \to \infty} \frac{2x^3 - x - 3}{6x^2 - x - 1}$

Practice Problems
Exercises 4.5, Page 245, 13 – 18, 19, 21, 23, 25, 27, 29, 31, 33, 37, 51 – 54

Practice Problems:
Review Worksheet 1 on Limits